

Riesz transforms associated with Schrödinger operators acting on weighted Hardy spaces

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator acting on $L^2(\mathbb{R}^n)$, $n \geq 1$, where $V \not\equiv 0$ is a nonnegative locally integrable function on \mathbb{R}^n . In this article, we will introduce weighted Hardy spaces $H_L^p(w)$ associated with L by means of the area integral function and study their atomic decomposition theory. We also show that the Riesz transform $\nabla L^{-1/2}$ associated with L is bounded from our new space $H_L^p(w)$ to the classical weighted Hardy space $H^p(w)$ when $\frac{n}{n+1} < p < 1$ and $w \in A_1 \cap RH_{(2/p)'}'$.
MSC: 35J10; 42B20; 42B30

Keywords: Weighted Hardy spaces; Riesz transform; Schrödinger operator; atomic decomposition; A_p weights

1 Introduction

Let $n \geq 1$ and V be a nonnegative locally integrable function defined on \mathbb{R}^n , not identically zero. We define the form \mathcal{Q} by

$$\mathcal{Q}(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^n} Vuv \, dx$$

with domain $\mathcal{D}(\mathcal{Q}) = \mathcal{V} \times \mathcal{V}$ where

$$\mathcal{V} = \{u \in L^2(\mathbb{R}^n) : \frac{\partial u}{\partial x_k} \in L^2(\mathbb{R}^n) \text{ for } k = 1, \dots, n \text{ and } \sqrt{V}u \in L^2(\mathbb{R}^n)\}.$$

It is well known that this symmetric form is closed. Note also that it was shown by Simon [17] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$ (the space of

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C^∞ functions with compact supports). In other words, $C_0^\infty(\mathbb{R}^n)$ is a core of the form \mathcal{Q} .

Let us denote by L the self-adjoint operator associated with \mathcal{Q} . The domain of L is given by

$$\mathcal{D}(L) = \{u \in \mathcal{D}(\mathcal{Q}) : \exists v \in L^2 \text{ such that } \mathcal{Q}(u, \varphi) = \int_{\mathbb{R}^n} v \varphi dx, \forall \varphi \in \mathcal{D}(\mathcal{Q})\}.$$

Formally, we write $L = -\Delta + V$ as a Schrödinger operator with potential V . Let $\{e^{-tL}\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and $p_t(x, y)$ be their kernels. Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \leq p_t(x, y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \quad (1.1)$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$.

The operator $\nabla L^{-1/2}$ is called the Riesz transform associated with L , which is defined by

$$\nabla L^{-1/2}(f)(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{-tL}(f)(x) \frac{dt}{\sqrt{t}}. \quad (1.2)$$

This operator is bounded on $L^2(\mathbb{R}^n)$ (see [11]). Moreover, it was proved in [1,3] that by using the molecular decomposition of functions in the Hardy space $H_L^1(\mathbb{R}^n)$, the operator $\nabla L^{-1/2}$ is bounded from $H_L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, and hence, by interpolation, is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$. Now assume that $V \in RH_q$ (Reverse Hölder class). In [15], Shen showed that $\nabla L^{-1/2}$ is a Calderón-Zygmund operator if $q \geq n$. When $\frac{n}{2} \leq q < n$, $\nabla L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq p_0$, where $1/p_0 = 1/q - 1/n$, and the above range of p is optimal. For more information about the Hardy spaces $H_L^p(\mathbb{R}^n)$ associated with Schrödinger operators for $0 < p \leq 1$, we refer the readers to [4,5,6].

In [18], Song and Yan introduced the weighted Hardy spaces $H_L^1(w)$ associated to L in terms of the area integral function and established their atomic decomposition theory. In the meantime, they also showed that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^p(w)$ for $1 < p < 2$, and bounded from $H_L^1(w)$ to the classical weighted Hardy space $H^1(w)$.

As a continuation of [18], the main purpose of this paper is to define the weighted Hardy spaces $H_L^p(w)$ associated to L for $0 < p < 1$ and study their atomic characterizations. We also obtain that $\nabla L^{-1/2}$ is bounded from $H_L^p(w)$ to the classical weighted Hardy space $H^p(w)$ for $\frac{n}{n+1} < p < 1$. Our main result is stated as follows.

Theorem 1.1. *Suppose that $L = -\Delta + V$. Let $\frac{n}{n+1} < p < 1$ and $w \in A_1 \cap RH_{(2/p)'}$. Then the operator $\nabla L^{-1/2}$ is bounded from $H_L^p(w)$ to the classical weighted Hardy space $HP(w)$.*

It is worth pointing out that when $L = -\Delta$ is the Laplace operator on \mathbb{R}^n , then the space $H_L^p(w)$ coincides with the classical weighted Hardy space $HP(w)$. Therefore, in this particular case, we derive that the classical Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded on $HP(w)$ for $\frac{n}{n+1} < p < 1$, which was already obtained by Lee and Lin in [12].

2 Notations and preliminaries

First, let us recall some standard definitions and notations. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [13]. A weight w is a locally integrable function on \mathbb{R}^n which takes values in $(0, \infty)$ almost everywhere, $B = B(x_0, r)$ denotes the ball with the center x_0 and radius r . We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n,$$

where C is a positive constant which is independent of B .

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \cdot \operatorname{ess\,inf}_{x \in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

A weight function w is said to belong to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left(\frac{1}{|B|} \int_B w(x)^r dx \right)^{1/r} \leq C \left(\frac{1}{|B|} \int_B w(x) dx \right) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

If $w \in A_p$ with $1 < p < \infty$, then we have $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. It follows from Hölder's inequality that $w \in RH_r$ implies $w \in RH_s$ for all $1 < s < r$. Moreover, if $w \in RH_r$, $r > 1$, then we have $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$.

Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B . For a given weight function w , we

denote the Lebesgue measure of B by $|B|$ and the weighted measure of B by $w(B)$, where $w(B) = \int_B w(x) dx$.

We give the following results that we will use in the sequel.

Lemma 2.1 ([8]). *Let $w \in A_p$, $p \geq 1$. Then, for any ball B , there exists an absolute constant C such that*

$$w(2B) \leq C w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C \cdot \lambda^{np} w(B),$$

where C does not depend on B nor on λ .

Lemma 2.2 ([8,9]). *Let $w \in A_p \cap RH_r$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \left(\frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left(\frac{|E|}{|B|} \right)^{(r-1)/r}$$

for any measurable subset E of a ball B .

Given a Muckenhoupt's weight function w on \mathbb{R}^n , for $0 < p < \infty$, we denote by $L^p(w)$ the space of all functions satisfying

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{A}{B} \leq C$. Moreover, we denote the conjugate exponent of $q > 1$ by $q' = q/(q-1)$.

3 Weighted Hardy spaces $H_L^p(w)$ for $0 < p < 1$ and their atomic decompositions

Let $L = -\Delta + V$. For any $t > 0$, we define $P_t = e^{-tL}$ and

$$Q_{t,k} = (-t)^k \frac{d^k P_s}{ds^k} \Big|_{s=t} = (tL)^k e^{-tL}, \quad k = 1, 2, \dots$$

We denote simply by Q_t when $k = 1$. First note that Gaussian upper bounds carry over from heat kernels to their time derivatives.

Lemma 3.1 ([2,14]). *For every $k = 1, 2, \dots$, there exist two positive constants C_k and c_k such that the kernel $p_{t,k}(x, y)$ of the operator $Q_{t,k}$ satisfies*

$$|p_{t,k}(x, y)| \leq \frac{C_k}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{c_k t}}$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^n$.

Set

$$H^2(\mathbb{R}^n) = \overline{\mathcal{R}(L)} = \overline{\{Lu \in L^2(\mathbb{R}^n) : u \in L^2(\mathbb{R}^n)\}},$$

where $\overline{\mathcal{R}(L)}$ stands for the range of L . We also set

$$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}.$$

For a given function $f \in L^2(\mathbb{R}^n)$, we consider the area integral function associated to Schrödinger operator L

$$S_L(f)(x) = \left(\iint_{\Gamma(x)} |Q_{t^2}(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Given a weight function w on \mathbb{R}^n , for $0 < p < 1$, we shall define the weighted Hardy spaces $H_L^p(w)$ as the completion of $H^2(\mathbb{R}^n)$ in the norm given by the $L^p(w)$ -norm of area integral function; that is

$$\|f\|_{H_L^p(w)} = \|S_L(f)\|_{L^p(w)}.$$

Let $M \in \mathbb{N}$ and $0 < p < 1$. As in [18], we say that a function $a(x) \in L^2(\mathbb{R}^n)$ is called a (p, M) -atom with respect to w (or a w -(p, M)-atom) if there exist a ball $B = B(x_0, r)$ and a function $b \in \mathcal{D}(L^M)$ such that

- (a) $a = L^M b$;
- (b) $\text{supp } L^k b \subseteq B, \quad k = 0, 1, \dots, M$;
- (c) $\|(r^2 L)^k b\|_{L^2(B)} \leq r^{2M} |B|^{1/2} w(B)^{-1/p}, \quad k = 0, 1, \dots, M$.

Let $M \in \mathbb{N}$ and $\frac{n}{n+1} < p < 1$. For any w -(p, M)-atom a associated to a ball $B = B(x_0, r)$, $\|a\|_{L^2(B)} \leq |B|^{1/2} w(B)^{-1/p}$, we will show that $a \in H_L^p(w)$ and its $H_L^p(w)$ -norm is uniformly bounded; precisely

Theorem 3.2. *Let $M \in \mathbb{N}$, $\frac{n}{n+1} < p < 1$ and $w \in A_1 \cap RH_{(2/p)'}'$. Then there exists a constant $C > 0$ independent of a such that*

$$\|S_L(a)\|_{L^p(w)} \leq C.$$

Proof. We write

$$\begin{aligned}\|S_L(a)\|_{L^p(w)}^p &= \int_{2B} |S_L(a)(x)|^p w(x) dx + \int_{(2B)^c} |S_L(a)(x)|^p w(x) dx \\ &= I_1 + I_2.\end{aligned}$$

Set $q = 2/p$. Note that $w \in RH_{q'}$, then it follows from Hölder's inequality, Lemma 2.1 and the L^2 boundedness of S_L (see (3.2) below) that

$$\begin{aligned}I_1 &\leq \left(\int_{2B} |S_L(a)(x)|^2 dx \right)^{p/2} \left(\int_{2B} w(x)^{q'} dx \right)^{1/q'} \\ &\leq C \|a\|_{L^2(B)}^p \cdot \frac{w(2B)}{|2B|^{1/q}} \\ &\leq C.\end{aligned}$$

We turn to deal with I_2 . By using Hölder's inequality and the fact that $w \in RH_{q'}$, we can get

$$\begin{aligned}I_2 &= \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |S_L(a)(x)|^p w(x) dx \\ &\leq C \sum_{k=1}^{\infty} \left(\int_{2^{k+1}B \setminus 2^k B} |S_L(a)(x)|^2 dx \right)^{p/2} \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/q}}.\end{aligned}$$

For any $x \in 2^{k+1}B \setminus 2^k B$, $k = 1, 2, \dots$, we write

$$\begin{aligned}&|S_L(a)(x)|^2 \\ &= \int_0^r \int_{|y-x|<t} |t^2 L e^{-t^2 L} a(y)|^2 \frac{dy dt}{t^{n+1}} + \int_r^\infty \int_{|y-x|<t} |t^2 L e^{-t^2 L} a(y)|^2 \frac{dy dt}{t^{n+1}} \\ &= \text{I} + \text{II}.\end{aligned}$$

For the term I, note that $0 < t < r$. By a simple calculation, we obtain that for any $(y, t) \in \Gamma(x)$, $x \in 2^{k+1}B \setminus 2^k B$, $z \in B$, then $|y - z| \geq 2^{k-1}r$. Hence, by using Hölder's inequality and Lemma 3.1, we deduce

$$\begin{aligned}|t^2 L e^{-t^2 L} a(y)| &\leq C \cdot \frac{t}{(2^{k-1}r)^{n+1}} \int_B |a(z)| dz \\ &\leq C \cdot \frac{t}{(2^k r)^{n+1}} \|a\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \\ &\leq C \cdot w(B)^{-1/p} \frac{t}{2^{k(n+1)} \cdot r}.\end{aligned}$$

Consequently

$$\begin{aligned} \text{I} &\leq C \left(\frac{1}{2^{k(n+1)} w(B)^{1/p}} \right)^2 \cdot \frac{1}{r^2} \int_0^r t \, dt \\ &\leq C \left(\frac{1}{2^{k(n+1)} w(B)^{1/p}} \right)^2. \end{aligned}$$

We now estimate the other term II. In this case, a direct computation shows that for any $(y, t) \in \Gamma(x)$, $x \in 2^{k+1}B \setminus 2^k B$ and $z \in B$, we have $t + |y - z| \geq 2^{k-1}r$. Since there exists a function $b \in \mathcal{D}(L^M)$ such that $a = L^M b$, then by Hölder's inequality and Lemma 3.1 again, we get

$$\begin{aligned} |t^2 L e^{-t^2 L} a(y)| &= |(t^2 L)^{M+1} e^{-t^2 L} b(y)| \cdot \frac{1}{t^{2M}} \\ &\leq C \cdot \frac{1}{(2^{k-1}r)^{n+1}} \int_B |b(z)| \, dz \cdot \frac{1}{t^{2M-1}} \\ &\leq C \cdot \frac{1}{(2^k r)^{n+1}} \|b\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \cdot \frac{1}{t^{2M-1}} \\ &\leq C \cdot \frac{r^{2M-1}}{2^{k(n+1)} w(B)^{1/p}} \cdot \frac{1}{t^{2M-1}}. \end{aligned}$$

Therefore

$$\begin{aligned} \text{II} &\leq C \left(\frac{1}{2^{k(n+1)} w(B)^{1/p}} \right)^2 \cdot r^{4M-2} \int_r^\infty \frac{dt}{t^{4M-1}} \\ &\leq C \left(\frac{1}{2^{k(n+1)} w(B)^{1/p}} \right)^2. \end{aligned}$$

Combining the above estimates for I and II, we thus obtain

$$|S_L(a)(x)| \leq C \cdot \frac{1}{2^{k(n+1)} w(B)^{1/p}}, \quad \text{when } x \in 2^{k+1}B \setminus 2^k B.$$

Then it follows immediately from Lemma 2.1 that

$$\begin{aligned} I_2 &\leq C \sum_{k=1}^{\infty} \frac{1}{2^{kp(n+1)} w(B)} \cdot w(2^{k+1}B) \\ &\leq C \sum_{k=1}^{\infty} \frac{1}{2^{kp(n+1)-kn}} \\ &\leq C, \end{aligned}$$

where in the last inequality we have used the fact that $p > n/(n+1)$. Summarizing the estimates for I_1 and I_2 derived above, we complete the proof of Theorem 3.2. \square

For every bounded Borel function $F : [0, \infty) \rightarrow \mathbb{C}$, we define the operator $F(L) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by the following formula

$$F(L) = \int_0^\infty F(\lambda) dE_L(\lambda),$$

where $E_L(\lambda)$ is the spectral decomposition of L . Therefore, the operator $\cos(t\sqrt{L})$ is well-defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [16] that there exists a constant c_0 such that the Schwartz kernel $K_{\cos(t\sqrt{L})}(x, y)$ of $\cos(t\sqrt{L})$ has support contained in $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq c_0 t\}$. By the functional calculus for L and Fourier inversion formula, whenever F is an even bounded Borel function with $\hat{F} \in L^1(\mathbb{R})$, we can write

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^\infty \hat{F}(t) \cos(t\sqrt{L}) dt.$$

Lemma 3.3 ([10]). *Let $\varphi \in C_0^\infty(\mathbb{R})$ be even and $\text{supp } \varphi \subseteq [-c_0^{-1}, c_0^{-1}]$. Let Φ denote the Fourier transform of φ . Then for each $j = 0, 1, \dots$, and for all $t > 0$, the Schwartz kernel $K_{(t^2 L)^j \Phi(t\sqrt{L})}(x, y)$ of $(t^2 L)^j \Phi(t\sqrt{L})$ satisfies*

$$\text{supp } K_{(t^2 L)^j \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$

For a given $s > 0$, we set

$$\mathcal{F}(s) = \left\{ \psi : \mathbb{C} \rightarrow \mathbb{C} \text{ measurable, } |\psi(z)| \leq C \frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

Then for any nonzero function $\psi \in \mathcal{F}(s)$, we have the following estimate (see [18])

$$\left(\int_0^\infty \|\psi(t\sqrt{L})f\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{1/2} \leq C \|f\|_{L^2(\mathbb{R}^n)}. \quad (3.1)$$

In particular, we have

$$\|S_L(f)\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}. \quad (3.2)$$

We are going to establish the atomic decomposition for the weighted Hardy spaces $H_L^p(w)$ ($0 < p < 1$).

Theorem 3.4. *Let $M \in \mathbb{N}$, $0 < p < 1$ and $w \in A_1$. If $f \in H_L^p(w)$, then there exist a family of w -(p, M)-atoms $\{a_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C \|f\|_{H_L^p(w)}^p$ such that f can be represented in the form $f(x) = \sum_j \lambda_j a_j(x)$, and the sum converges both in the sense of $L^2(\mathbb{R}^n)$ -norm and $H_L^p(w)$ -norm.*

Proof. First assume that $f \in H_L^p(w) \cap H^2(\mathbb{R}^n)$. We follow the same constructions as in [18]. Let φ and Φ be as in Lemma 3.3. We set $\Psi(x) = x^{2M}\Phi(x)$, $x \in \mathbb{R}$. By the L^2 -functional calculus of L , for every $f \in H^2(\mathbb{R}^n)$, we can establish the following version of the Calderón reproducing formula

$$f(x) = c_\psi \int_0^\infty \Psi(t\sqrt{L})t^2 L e^{-t^2 L}(f)(x) \frac{dt}{t}, \quad (3.3)$$

where the above equality holds in the sense of $L^2(\mathbb{R}^n)$ -norm. For any $k \in \mathbb{Z}$, set

$$\Omega_k = \{x \in \mathbb{R}^n : S_{L,10\sqrt{n}}(f)(x) > 2^k\}.$$

Let \mathbb{D} denote the set formed by all dyadic cubes in \mathbb{R}^n and let

$$\mathbb{D}_k = \left\{ Q \in \mathbb{D} : w(Q \cap \Omega_k) > \frac{w(Q)}{2}, w(Q \cap \Omega_{k+1}) \leq \frac{w(Q)}{2} \right\}.$$

Obviously, for any $Q \in \mathbb{D}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathbb{D}_k$. We also denote the maximal dyadic cubes in \mathbb{D}_k by Q_k^l . Note that the maximal dyadic cubes Q_k^l are pairwise disjoint, then it is easy to check that

$$\sum_l w(Q_k^l) \leq C \cdot w(\Omega_k). \quad (3.4)$$

Set

$$\tilde{Q} = \left\{ (y, t) \in \mathbb{R}_+^{n+1} : y \in Q, \frac{l(Q)}{2} < t \leq l(Q) \right\},$$

where $l(Q)$ denotes the side length of Q . If we set $\tilde{Q}_k^l = \bigcup_{Q_k^l \supseteq Q \in \mathbb{D}_k} \tilde{Q}$, then we have $\mathbb{R}_+^{n+1} = \bigcup_k \bigcup_l \tilde{Q}_k^l$. Hence, by the formula (3.3), we can write

$$f(x) = \sum_k \sum_l c_\psi \int_{\tilde{Q}_k^l} \Psi(t\sqrt{L})(x, y) t^2 L e^{-t^2 L} f(y) \frac{dy dt}{t} = \sum_k \sum_l \lambda_{kl} a_k^l(x),$$

where $a_k^l = L^M b_k^l$,

$$b_k^l(x) = c_\psi \lambda_{kl}^{-1} \int_{\tilde{Q}_k^l} t^{2M} \Phi(t\sqrt{L})(x, y) t^2 L e^{-t^2 L} f(y) \frac{dy dt}{t}$$

and

$$\lambda_{kl} = w(Q_k^l)^{1/p-1/2} \left(\int_{\tilde{Q}_k^l} |t^2 L e^{-t^2 L} f(y)|^2 \frac{w(Q_k^l)}{|Q_k^l|} \frac{dy dt}{t} \right)^{1/2}.$$

By using Lemma 3.3, the authors in [18] showed that for every $j = 0, 1, \dots, M$, $\text{supp}(L^j b_k^l) \subseteq 3Q_k^l$. In [18], they also obtained the following estimate

$$\sum_l \int_{\widetilde{Q_k^l}} |t^2 L e^{-t^2 L} f(y)|^2 \frac{w(Q_k^l)}{|Q_k^l|} \frac{dy dt}{t} \leq C \cdot 2^{2k} w(\Omega_k). \quad (3.5)$$

Since

$$\|(l(Q_k^l)^2 L)^j b_k^l\|_{L^2(3Q_k^l)} = \sup_{\|h\|_{L^2(3Q_k^l)} \leq 1} \left| \int_{\mathbb{R}^n} (l(Q_k^l)^2 L)^j b_k^l(x) h(x) dx \right|.$$

Let $\Psi_j(x) = x^{2j} \Phi(x)$, $j = 0, 1, \dots, M$. Then we can easily verify that $\Psi_j \in \mathcal{F}(2j)$. Observe that when $(y, t) \in \widetilde{Q_k^l}$, we have $t \sim l(Q_k^l)$. Then it follows from Hölder's inequality and the estimate (3.1) that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} (l(Q_k^l)^2 L)^j b_k^l(x) h(x) dx \right| \\ & \leq \frac{C \cdot l(Q_k^l)^{2M}}{\lambda_{kl}} \left(\int_{\widetilde{Q_k^l}} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy dt}{t} \right)^{1/2} \\ & \quad \times \left(\int_{\widetilde{Q_k^l}} \left| (t^2 L)^j \Phi(t\sqrt{L})(h\chi_{3Q_k^l})(y) \right|^2 \frac{dy dt}{t} \right)^{1/2} \\ & \leq C \cdot l(Q_k^l)^{2M} |Q_k^l|^{1/2} w(Q_k^l)^{-1/p} \left(\int_0^\infty \|\Psi_j(t\sqrt{L})(h\chi_{3Q_k^l})\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{1/2} \\ & \leq C \cdot l(Q_k^l)^{2M} |Q_k^l|^{1/2} w(Q_k^l)^{-1/p}. \end{aligned}$$

Hence

$$\|(l(Q_k^l)^2 L)^j b_k^l\|_{L^2(3Q_k^l)} \leq C \cdot l(Q_k^l)^{2M} |Q_k^l|^{1/2} w(Q_k^l)^{-1/p}.$$

From the above discussions, we have proved that these functions a_k^l are all w -(p, M)-atoms up to a normalization by a multiplicative constant. Finally, by using Hölder's inequality, the estimates (3.4) and (3.5), we obtain

$$\begin{aligned} \sum_k \sum_l |\lambda_{kl}|^p &= \sum_k \sum_l \left(w(Q_k^l) \right)^{1-p/2} \left(\int_{\widetilde{Q_k^l}} |t^2 L e^{-t^2 L} f(y)|^2 \frac{w(Q_k^l)}{|Q_k^l|} \frac{dy dt}{t} \right)^{p/2} \\ &\leq \sum_k \left(\sum_l w(Q_k^l) \right)^{1-p/2} \left(\sum_l \int_{\widetilde{Q_k^l}} |t^2 L e^{-t^2 L} f(y)|^2 \frac{w(Q_k^l)}{|Q_k^l|} \frac{dy dt}{t} \right)^{p/2} \\ &\leq C \sum_k \left(w(\Omega_k) \right)^{1-p/2} \left(2^{2k} w(\Omega_k) \right)^{p/2} \\ &\leq C \|S_L(f)\|_{L^p(w)}^p. \end{aligned}$$

Therefore, we have established the atomic decomposition for all functions in the space $H_L^p(w) \cap H^2(\mathbb{R}^n)$. By a standard density argument, we can show that the same conclusion holds for $H_L^p(w)$. Following along the same arguments as in [18], we can also prove that the sum $f = \sum_j \lambda_j a_j$ converges both in the sense of $L^2(\mathbb{R}^n)$ -norm and $H_L^p(w)$ -norm, the details are omitted here. This completes the proof of Theorem 3.4. \square

4 Proof of Theorem 1.1

We shall need the following Davies-Gaffney estimate which can be found in [10,18].

Lemma 4.1. *For any two closed sets E and F of \mathbb{R}^n , there exist two positive constants C and c such that*

$$\|t\nabla e^{-t^2 L} f\|_{L^2(F)} \leq C \cdot e^{-\frac{\text{dist}(E,F)^2}{ct^2}} \|f\|_{L^2(E)}$$

for every $f \in L^2(\mathbb{R}^n)$ with support contained in E .

Theorem 4.2. *Let $\frac{n}{n+1} < p < 1$ and $w \in A_1 \cap RH_{(2/p)'}'$. Then the operator $\nabla L^{-1/2}$ is bounded from $H_L^p(w)$ to $L^p(w)$.*

Proof. By Theorem 3.4 we just proved, it is enough for us to show that for any w -(p, M)-atom a , $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, there exists a constant $C > 0$ independent of a such that $\|\nabla L^{-1/2}(a)\|_{L^p(w)} \leq C$. Let a be a w -(p, M)-atom with $\text{supp } a \subseteq B = B(x_0, r)$, $\|a\|_{L^2(B)} \leq |B|^{1/2} w(B)^{-1/p}$. We write

$$\begin{aligned} \|\nabla L^{-1/2}(a)\|_{L^p(w)}^p &= \int_{2B} |\nabla L^{-1/2}(a)(x)|^p w(x) dx + \int_{(2B)^c} |\nabla L^{-1/2}(a)(x)|^p w(x) dx \\ &= J_1 + J_2. \end{aligned}$$

Set $q = 2/p$. Applying Hölder's inequality, the L^2 boundedness of $\nabla L^{-1/2}$, Lemma 2.1 and $w \in RH_{q'}$, we thus have

$$\begin{aligned} J_1 &\leq \left(\int_{2B} |\nabla L^{-1/2}(a)(x)|^2 dx \right)^{p/2} \left(\int_{2B} w(x)^{q'} dx \right)^{1/q'} \\ &\leq C \|a\|_{L^2(B)}^p \cdot \frac{w(2B)}{|2B|^{1/q}} \\ &\leq C. \end{aligned}$$

On the other hand, it follows from Hölder's inequality and $w \in RH_{q'}$ that

$$\begin{aligned} J_2 &= \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^p w(x) dx \\ &\leq C \sum_{k=1}^{\infty} \left(\int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^2 dx \right)^{p/2} \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/q}}. \end{aligned} \quad (4.1)$$

By a change of variable $s = t^2$, we can rewrite (1.2) as

$$\nabla L^{-1/2}(a)(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} s \nabla e^{-s^2 L}(a)(x) \frac{ds}{s}. \quad (4.2)$$

For any $k = 1, 2, \dots$, it follows immediately from Minkowski's integral inequality that

$$\begin{aligned} &\left(\int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^2 dx \right)^{1/2} \\ &\leq C \int_0^r \|s \nabla e^{-s^2 L} a\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s} + C \int_r^{\infty} \|s \nabla e^{-s^2 L} a\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s} \\ &= \text{III} + \text{IV}. \end{aligned}$$

Observe that $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Then we are able to choose a positive number N such that $\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) < N < M$. By using Lemma 4.1, we can get

$$\begin{aligned} \text{III} &\leq C \int_0^r e^{-\frac{(2^k r)^2}{s^2}} \|a\|_{L^2(B)} \frac{ds}{s} \\ &\leq C \int_0^r \frac{s^{2N}}{(2^k r)^{2N}} \frac{ds}{s} \cdot \|a\|_{L^2(B)} \\ &\leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p}. \end{aligned} \quad (4.3)$$

We now turn to estimate the term IV. Since $a = L^M b$ and $\|b\|_{L^2(B)} \leq r^{2M} |B|^{1/2} w(B)^{-1/p}$. Using Lemma 3.1 and Lemma 4.1, we deduce

$$\begin{aligned} \text{IV} &= C \int_r^{\infty} \|s \nabla e^{-s^2 L}(L^M b)\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s} \\ &= C \int_r^{\infty} \|s \nabla e^{-\frac{s^2 L}{2}} (s^2 L)^M e^{-\frac{s^2 L}{2}} b\|_{L^2(2^{k+1}B \setminus 2^k B)} \frac{ds}{s^{2M+1}} \\ &\leq C \int_r^{\infty} e^{-\frac{(2^k r)^2}{s^2}} \|(s^2 L)^M e^{-\frac{s^2 L}{2}} b\|_{L^2(B)} \frac{ds}{s^{2M+1}} \\ &\leq C \int_r^{\infty} \frac{s^{2N}}{(2^k r)^{2N}} \frac{ds}{s^{2M+1}} \cdot \|b\|_{L^2(B)} \\ &\leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p}. \end{aligned} \quad (4.4)$$

Combining the above inequality (4.4) with (4.3), we thus obtain

$$\left(\int_{2^{k+1}B \setminus 2^k B} |\nabla L^{-1/2}(a)(x)|^2 dx \right)^{1/2} \leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p}. \quad (4.5)$$

Substituting the above inequality (4.5) into (4.1) and using Lemma 2.1, then we have

$$\begin{aligned} J_2 &\leq C \sum_{k=1}^{\infty} \left(2^{-2kN} |B|^{1/2} w(B)^{-1/p} \right)^p \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{p/2}} \\ &\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k(2pN + \frac{np}{2} - n)}} \\ &\leq C, \end{aligned}$$

where the last series is convergent since $N > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Summarizing the estimates for J_1 and J_2 , we get the desired result. \square

The real-variable theory of classical weighted Hardy spaces have been extensively studied by many authors. In 1979, Garcia-Cuerva studied the atomic decomposition and the dual spaces of $H^p(w)$ for $0 < p \leq 1$. In 2002, Lee and Lin gave the molecular characterization of $H^p(w)$ for $0 < p \leq 1$, they also obtained the $H^p(w)$ ($\frac{1}{2} < p \leq 1$) boundedness of the Hilbert transform and the $H^p(w)$ ($\frac{n}{n+1} < p \leq 1$) boundedness of the Riesz transforms. For the results mentioned above, we refer the readers to [7,12,19] for further details.

Let $\frac{n}{n+1} < p < 1$ and $w \in A_1$. A real-valued function $a(x)$ is called a w -($p, 2, 0$)-atom if the following conditions are satisfied (see [7,19]):

- (a) $\text{supp } a \subseteq B$;
- (b) $\|a\|_{L^2(B)} \leq |B|^{1/2} w(B)^{-1/p}$;
- (c) $\int_{\mathbb{R}^n} a(x) dx = 0$.

Theorem 4.3. *Let $\frac{n}{n+1} < p < 1$ and $w \in A_1$. For each $f \in H^p(w)$, there exist a family of w -($p, 2, 0$)-atoms $\{a_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C \|f\|_{H^p(w)}^p$ such that $f = \sum_j \lambda_j a_j$ in the sense of $H^p(w)$ norm.*

Next, as in [18], we shall also define the new weighted molecules for $H^p(w)$. Let $\frac{n}{n+1} < p < 1$, $w \in A_1$ and $\varepsilon > 0$. A function $m(x) \in L^2(\mathbb{R}^n)$ is called a w -($p, 2, 0, \varepsilon$)-molecule associated to a ball B if the following conditions are satisfied:

- (A) $\int_{\mathbb{R}^n} m(x) dx = 0$;
- (B) $\|m\|_{L^2(2B)} \leq |B|^{1/2} w(B)^{-1/p}$;

$$(C) \|m\|_{L^2(2^{k+1}B \setminus 2^k B)} \leq 2^{-k\varepsilon} |2^k B|^{1/2} w(2^k B)^{-1/p}, \quad k = 1, 2, \dots$$

Note that for every w -($p, 2, 0$)-atom, it is a w -($p, 2, 0, \varepsilon$)-molecule for all $\varepsilon > 0$. Then we are able to establish the following molecular characterization for the classical weighted Hardy spaces $H^p(w)$.

Theorem 4.4. *Let $\frac{n}{n+1} < p < 1$ and $w \in A_1$.*

(i) *If $f \in H^p(w)$, then there exist a family of w -($p, 2, 0, \varepsilon$)-molecules $\{m_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C \|f\|_{H^p(w)}^p$ such that $f = \sum_j \lambda_j a_j$ in the sense of $H^p(w)$ norm.*

(ii) *Suppose that $w \in RH_{(2/p)'}$ and $\varepsilon > n/2$, then every w -($p, 2, 0, \varepsilon$)-molecule m is in $H^p(w)$. Moreover, there exists a constant $C > 0$ independent of m such that $\|m\|_{H^p(w)} \leq C$.*

Proof. (i) is a straightforward consequence of Theorem 4.3.

(ii) We follow the idea of [18]. Denote $m_0(x) = m(x)\chi_{2B}(x)$, $m_k(x) = m(x)\chi_{2^{k+1}B \setminus 2^k B}(x)$, $k = 1, 2, \dots$. Then we can decompose $m(x)$ as

$$m(x) = \sum_{k=0}^{\infty} m_k(x) = \sum_{k=0}^{\infty} (m_k(x) - N_k(x)) + \sum_{k=0}^{\infty} N_k(x),$$

where $N_0(x) = \frac{1}{|2B|} \int_{\mathbb{R}^n} m_0(y) dy \cdot \chi_{2B}(x)$ and $N_k(x) = \frac{1}{|2^{k+1}B \setminus 2^k B|} \int_{\mathbb{R}^n} m_k(y) dy \cdot \chi_{2^{k+1}B \setminus 2^k B}(x)$, $k = 1, 2, \dots$. Following along the same lines as in [18], we can also show that each $(m_k - N_k)$ is a multiple of w -($p, 2, 0$)-atom with a sequence of coefficients in l^p . We set $\eta_k = \int_{\mathbb{R}^n} m_k(y) dy$, $k = 0, 1, \dots$. In [18], Song and Yan established the following identity

$$\sum_{k=0}^{\infty} N_k(x) = \sum_{k=0}^{\infty} p_k \cdot \psi_k(x),$$

where $p_k = \sum_{j=k+1}^{\infty} \eta_j$ and $\psi_k(x) = \frac{N_{k+1}(x)}{\eta_{k+1}} - \frac{N_k(x)}{\eta_k}$. Then we have

$$\begin{aligned} |p_k| &\leq \sum_{j=k+1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |m(y)| dy \\ &\leq \sum_{j=k+1}^{\infty} \|m\|_{L^2(2^{j+1}B \setminus 2^j B)} |2^{j+1}B|^{1/2} \\ &\leq C \sum_{j=k+1}^{\infty} 2^{-j\varepsilon} \cdot |2^j B| w(2^j B)^{-1/p}. \end{aligned}$$

When $j \geq k+1$, then $2^k B \subseteq 2^j B$. Since $w \in RH_{(2/p)'}'$, then by Lemma 2.2, we can get

$$\frac{w(2^k B)}{w(2^j B)} \leq C \left(\frac{|2^k B|}{|2^j B|} \right)^{p/2}.$$

Hence

$$\begin{aligned} |p_k| &\leq C \cdot \frac{|2^k B|}{w(2^k B)^{1/p}} \sum_{j=k+1}^{\infty} 2^{-j\varepsilon} \left(\frac{|2^j B|}{|2^k B|} \right)^{1/2} \\ &\leq C \cdot \frac{|2^k B|}{w(2^k B)^{1/p}} \left(\sum_{j=k+1}^{\infty} 2^{-j(\varepsilon-n/2)} \right) \cdot 2^{-kn/2} \\ &\leq C \cdot 2^{-k\varepsilon} \frac{|2^k B|}{w(2^k B)^{1/p}}, \end{aligned}$$

where the last inequality holds since $\varepsilon > n/2$. As in [18], we can easily check that $2^{k\varepsilon} p_k \psi(x)$ are all w -($p, 2, 0$)-atoms associated to $2^{k+1} B$. Therefore the sum $\sum_{k=0}^{\infty} N_k$ can be write as an infinite linear combination of w -($p, 2, 0$)-atoms with a sequence of coefficients in l^p . Summarizing the above discussions, we complete the proof of Theorem 4.4. \square

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.4 and Theorem 4.4, it suffices to show that for every w -(p, M)-atom a with $\text{supp } a \subseteq B$, then $\nabla L^{-1/2} a$ is a w -($p, 2, 0, \varepsilon$)-molecule, where $M > n(\frac{1}{p} - \frac{1}{2})$ and $\varepsilon > n/2$. It is easy to see that $\int_{\mathbb{R}^n} \nabla L^{-1/2} a(x) dx = 0$. It remains to verify the estimates (B) and (C). Hölder's inequality and the definition of w -(p, M)-atom imply

$$\|\nabla L^{-1/2}(a)\|_{L^2(2B)} \leq C \|a\|_{L^2(B)} \leq C \cdot |B|^{1/2} w(B)^{-1/p}.$$

For $k = 1, 2, \dots$, it follows from the previous estimates (4.3) and (4.4) that

$$\|\nabla L^{-1/2}(a)\|_{L^2(2^{k+1}B \setminus 2^k B)} \leq C \cdot 2^{-2kN} |B|^{1/2} w(B)^{-1/p},$$

where $N > 0$ is chosen such that $n(\frac{1}{p} - \frac{1}{2}) < N < M$. By using Lemma 2.2, we get

$$\frac{w(B)}{w(2^k B)} \geq C \cdot \frac{|B|}{|2^k B|}.$$

Hence

$$\|\nabla L^{-1/2}(a)\|_{L^2(2^{k+1}B \setminus 2^k B)} \leq C \cdot 2^{-k(2N-n/p+n/2)} |2^k B|^{1/2} w(2^k B)^{-1/p}.$$

Therefore, we have proved $\nabla L^{-1/2} a$ is a w -($p, 2, 0, 2N - n/p + n/2$)-molecule. This concludes the proof of Theorem 1.1. \square

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